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Time-Space Noncommutativity And Symmetries For A Massive Relativistic Particle

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Abstract: We show the existence of a time-space noncommutativity (NC) for the physical system of a massive relativistic particle by exploiting the underlying symmetry properties of this system. The space-space NC is eliminated by the consideration of the exact symmetry properties and their consistency with the equations of motion for the above system. The symmetry corresponding to the noncommutative geometry turns out to be the special case of the gauge symmetry such that the mass parameter of the above system becomes noncommutative with the space and time variables. The possible deformations of the gauge algebra between the spacetime variables and the angular momenta are discussed in detail. These modifications owe their origin to the NC of the mass parameter with the space and time variables. The cohomological origin for the above NC is addressed in the language of the off-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST) symmetry transformations.

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1 Introduction

The subject of noncommutative field theories has been a topic of intensive research activities during the past few years. Despite the fact that the idea of noncommutativity (NC) in the spacetime structure has a long history [1,2], the recent upsurge of interest in the NC (and the corresponding field theories constructed on the noncommutative spacetime) has been spurred due to its very clear and cogent appearance in the context of string theories, D-branes and M-theories. To be more precise , the end points of the strings, trapped on the D-branes, turn out to be noncommutative in the presence of a background two-form gauge field [3,4]. It has been argued, furthermore, that the string dynamics could be shown to be equivalent to the minimally coupled gauge field theory defined on a noncommutative space [5]. From a distinctly different point of view, a careful study of the quantum gravity and black hole physics entails upon the spacetime structure to become noncommutative in nature [6,7]. In other words, an attempt to unify the key concepts of quantum mechanics and gravity (which is a benchmark of theoretical physics at an energy scale comparable to the Planck energy) requires, by various considerations, a spacetime structure where the D-dimensional spacetime variables x_μ ($\mu = 0, 1, 2 ... D - 1$) become noncommutative (i.e. $[x_\mu, x_\nu] \neq 0$) leading to an uncertainty relation between the two of them.

It is evident from the above discussions that the impact of the NC in spacetime structure would manifest itself only at a very high energy scale. However, it might also be possible to physically test its very existence through the low-energy effective actions, obtained in some specific limits, from the actual high energy theory. This is why a few experiments have been suggested in the literature to observe the impact of these NCs on some physically interesting quantities. It has also been argued that only the lowest order quantum mechanical effects are good enough to shed some light on the spacetime NC in the context of celebrated Aharonov-Bohm type of experiments and/or synchrotron radiation studies [8-10]. However, so far, there have not been any successful observations of these effects. There is an alternative way to theoretically predict some consequences of the spacetime NC on some physically interesting quantities. In this approach, one can exploit the basic concepts of the noncommutative geometry (as well as the corresponding noncommutative field theories) and construct the low energy effective actions. In this category, one can mention a couple of attempts that are basically connected with the noncommutative Chern-Simons theory [11] and the noncommutative standard model of unification [12]. The physical consequences of these studies are, however, yet to be tested experimentally.

The reparametrization invariant models of the free as well as interacting relativistic (super)particles are at the heart of the modern developments in the (super)string theories and (super)gravity theories. These models are also found to be endowed with the first-class constraints in the language of the Dirac's prescription for the classification of constraints [13,14]. As a consequence, the above models respect a set of local gauge symmetry transformations that is generated by the first-class constraints themselves. Thus, the above models

clearly possess a whole host of interesting mathematical as well as physical structures. The purpose of our present paper is to study the model of the free massive relativistic particle and to demonstrate the existence of a specific variety of time-space NC in the spacetime structure (see, e.g., [15-17] for details) by exploiting the continuous symmetry properties of the Lagrangians for the above system. In particular, we lay stress on the standard- as well as non-standard gauge type of symmetry transformations (and their consistency with the equations of motion) which imply the presence of commutativity and NC in the theory. It is worthwhile to mention that a whole range of studies, connected with the unitary quantum mechanics and their physical consequences [15-17], have been performed based on the above type of NC. Thus, the above time-space NC is physically very interesting.

It is pertinent to point out that, in a couple of papers [18,19], the reparametrization invariant models and their mechanics have been studied where the equivalence of the NC and commutativity in the spacetime structure has been demonstrated in the language of the Dirac bracket formalism for two gauge choices. The deformation of the Poincaré algebra and the algebra between the spacetime variables and the angular momenta has been obtained due to the presence of the NC in spacetime. For the massless relativistic particle (that is endowed with more symmetry properties than its massive counterpart), an extension of the total conformal algebra has been shown to exist due to the NC in spacetime [20]. The latter emerges due to the presence of a *new* scale type of symmetry (that is different from the global scale symmetry of the usual conformal group of spacetime transformations). The dynamical implications of the above NC have been studied thoroughly in the language of the symplectic structure and Poisson bracket formalism [21]. In a very recent paper [22], a toy model of the reparametrization invariant free non-relativistic particle (see, e.g., [19]) has been studied in detail where the equivalence between the NC and commutativity has been demonstrated in the language of the standard- and non-standard gauge type of symmetry transformations for the Lagrangian of the system. As it turns out, the symmetry corresponding to the noncommutative geometry entails upon the mass parameter of this model to become noncommutative in nature with the space variable *alone*.

In our present paper, we generalize our earlier idea [22], to the reparametrization invariant model of a free massive relativistic particle and show the existence of the NC in spacetime structure by exploiting the continuous symmetry properties of the Lagrangian for the system. It should be emphasized that we obtain a unique symmetry transformation (see, e.g., (3.10) below) from the standard gauge symmetry transformations (corresponding to a commutative geometry) as well as from the non-standard gauge-type of symmetry transformations (corresponding to a noncommutative geometry). The important point, to be noted, is the fact that one can *not* guess the unique symmetry transformations (see, e,g, (3.10) below) from the standard gauge symmetry transformations. Rather, it is logically derived from the non-standard gauge type of transformations (corresponding to a noncommutative geometry) by requiring the consistency between the non-standard symmetry transformations and the equations of motion derived from the equivalent Lagrangians for

the system. In the above consistency requirement, the expressions for the canonical momenta, obtained from the different (but equivalent) Lagrangians, play very important role. It is clear that the unique symmetry transformation (see, e.g., (3.10) below) entails upon (i) the spacetime to become commutative from one point of view (i.e. standard gauge transformations), and (ii) the spacetime to become noncommutative from another point of view (i.e. the non-standard gauge type transformations). This establishes a couple of important results in one stroke. First, it demonstrates the equivalence of the commutativity and NC in the theory which agrees with such an observation made in the language of the Dirac bracket formalism (see, e.g., [18,19] for details). Second, the symmetry corresponding to the noncommutative geometry enforces the mass parameter to become noncommutative with *both* the space and time variables (see, e.g., (3.12) below) which is *different* from our earlier result in [22]. We also study, in our present paper, the deformation of the Poincaré algebra and the algebra between the spacetime variables and the angular momenta due to the NC in spacetime structure brought in by the non-standard gauge type transformations (and the NC of the mass parameter with both the space and time variables). We provide a logical basis for the existence of the unique symmetry transformation (see, e.g., (3.10)) for the model in the language of the local, continuous and nilpotent Becchi-Rouet-Stora-Tyutin (BRST) symmetry transformations and the corresponding cohomology.

Our present study is essential primarily on three counts. First and foremost, it is a very nice and straightforward generalization of the idea proposed in [22] (for the discussion of the toy model of a reparametrization invariant non-relativistic free particle) to the physically interesting reparametrization invariant model of a free massive relativistic particle. Second, the specific type of the time-space NC (i.e. $\{X_0, X_i\}_{(PB)} \neq 0, \{X_i, X_j\}_{(PB)} = 0$ because $\theta_{0i} \neq 0, \theta_{ij} = 0$) emerges naturally in our present endeavour (cf. (3.1) below) from the symmetry considerations. This type of NC is not chosen *ab initio* in our present discussion as is the case in [15-17]. Finally, the NC and commutativity in spacetime structure are shown to be the distinct and different aspects of a unique set of symmetry transformations (cf. (3.10) below) for the dynamical variables of the Lagrangian. The above unique symmetry transformations are derived from the non-standard gauge-type symmetry transformations (corresponding to a noncommutative geometry) and it is almost impossible to guess them from the standard gauge symmetry transformations (corresponding to a commutative geometry). It appears to be an interesting coincidence that these unique symmetry transformations happen to be a particular case of the standard gauge transformations.

The outline of our present paper is as follows. In Section 2, we recapitulate the bare essentials of the continuous symmetry transformations respected by the Lagrangians for the free massive relativistic particle. Section 3 is devoted to the discussion of the non-standard gauge-type symmetry transformations for the spacetime variables which lead to the presence of a NC in the spacetime structure. We discuss the deformation of the gauge algebra between the angular momentum generator and spacetime variables in Section 4. Minor modification of the Poincaré algebra is also briefly sketched in this section. We provide a

cohomological origin, in Section 5, for the NC of the spacetime structure in the language of BRST symmetry transformations. Finally, in Section 6, we make some concluding remarks and point out a few future directions for further investigations.

2 Preliminary: Standard Symmetries and Commutativity

We begin with a set of three different looking (but equivalent) Lagrangians for the free massive relativistic particle, moving on a trajectory that is embedded in a D-dimensional flat Minkowskian target space †. These Lagrangians are (see, e.g., [23,24]):

$$L_0 = m\,(\dot{x}^2)^{1/2}, \quad L_f = p_\mu \dot{x}^\mu - \frac{1}{2}\,e\,(p^2 - m^2), \quad L_s = \frac{1}{2}\,\frac{\dot{x}^2}{e} + \frac{1}{2}\,e\,m^2. \tag{2.1}$$

Some of the key common features of the above Lagrangians are (i) the force free ($\dot{p}_\mu = 0$) motion, (ii) the mass-shell condition ($p^2 - m^2 = 0$), and (iii) the reparametrization invariance under $\tau \to \tau' = f(\tau)$ where τ is the parameter that characterizes the trajectory (i.e. the world-line) of the relativistic particle in the target space and $f(\tau)$ is any arbitrary well-defined function of τ. Except for the mass (i.e. the analogue of the cosmological) parameter m, the canonically conjugate target space phase variables (i.e. $x^\mu(\tau),\ p_\mu(\tau)$) and the einbein field $e(\tau)$ are functions of the monotonically increasing parameter τ and $\dot{x}^\mu = (dx^\mu/d\tau)$, etc. It is clear from the Lagrangian L_0 (with the square root) and the second-order Lagrangian L_s (with e in the denominator) that the following expressions

$$p_\mu = \frac{m \dot{x}_\mu}{(\dot{x}^2)^{1/2}} \;\equiv\; \frac{m \dot{x}_\mu}{[\dot{x}_0^2 - \dot{x}_i^2]^{1/2}}, \qquad e = \frac{(\dot{x}^2)^{1/2}}{m} \;\equiv\; \frac{[\dot{x}_0^2 - \dot{x}_i^2]^{1/2}}{m}, \tag{2.2}$$

are correct. With the above inputs, it is evident that the first-order Lagrangian L_f, re-expressed in the following long-hand form

$$L_f = p_0 \dot{x}_0 - p_i \dot{x}_i - \tfrac{1}{2}\,e\,(p_0^2 - p_i^2 - m^2), \tag{2.3}$$

is equivalent to the other Lagrangians L_0 and L_s. Even though the above Lagrangians are equivalent and mutually consistent with one-another, we shall be concentrating only on the first-order Lagrangian (2.3) for the rest of our discussions ‡. It can be checked that the following symmetry transformations (δ_r)

$$\delta_r x_0 = \epsilon \dot{x}_0, \quad \delta_r x_i = \epsilon \dot{x}_i, \quad \delta_r p_0 = \epsilon \dot{p}_0, \quad \delta_r p_i = \epsilon \dot{p}_i, \quad \delta_r e = \frac{d}{d\tau}\,[\epsilon e], \tag{2.4}$$

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†The flat Minkowskian D-dimensional target spacetime manifold, in our adopted convention and notations, is characterized by the metric $\eta_{\mu\nu} = \text{diag}\,(+1, -1, -1, ...)$ so that the scalar product $(A \cdot B)$ between two non-null vectors A^μ and B^μ is defined as $(A \cdot B) = \eta_{\mu\nu} A^\mu B^\nu \equiv \eta^{\mu\nu} A_\mu B_\nu = A_0 B_0 - A_i B_i \equiv A_\mu B^\mu$. Here the Greek indices $\mu, \nu, \lambda..... = 0, 1, 2, ...D-1$ correspond to the time and space directions on the manifold and the Latin indices $i, j, k..... = 1, 2, ...D-1$ stand for the space directions only.

‡The first-order Lagrangian L_f is simpler in the sense that (i) it is without any square root, and (ii) it is devoid of any variables (and their derivative(s) w.r.t. τ) in the denominator. In addition, it is endowed with the largest number of dynamical variables (x_0, x_i, p_0, p_i, e). The latter feature of L_f allows more freedom for theoretical discussions (connected with it) than the other two Lagrangians L_0 and L_s.

generated due to the infinitesimal reparametrization transformation $\tau \rightarrow \tau' = \tau - \epsilon(\tau)$, leave the Lagrangian (2.3) quasi-invariant because $\delta_r L_f = (d/d\tau)[\epsilon L_f]$. Here $\epsilon(\tau)$ is an infinitesimal local parameter for the reparametrization transformation and $\delta_r \phi(\tau) = \phi'(\tau) - \phi(\tau)$ for the generic field $\phi = x_0, x_i, p_0, p_i, e$. Furthermore, it is evident that the first-order Lagrangian (2.3) is endowed with a couple of first-class constraints ($\Pi_e \approx 0, p_0^2 - p_i^2 - m^2 \approx 0$) in the language of Dirac's prescription for the classification of constraints. Here Π_e is the conjugate momentum corresponding to the einbein field $e(\tau)$. These constraints generate [§] the following gauge symmetry (δ_g) transformations (with the infinitesimal parameter ξ)

$$\delta_g x_0 = \xi p_0, \quad \delta_g x_i = \xi p_i, \quad \delta_g p_0 = 0, \quad \delta_g p_i = 0, \quad \delta_g e = \dot{\xi}, \tag{2.5}$$

under which the Lagrangian L_f remains quasi-invariant because $\delta_g L_f = (d/d\tau)[(\xi/2)(p_0^2 - p_i^2 + m^2)]$. It is clear that the gauge symmetry transformations (2.5) and the infinitesimal reparametrization transformations (2.4) are equivalent for (i) the identification $\xi = \epsilon e$, and (ii) the validity of the equations of motion $\dot{x}_0 = e p_0, \dot{x}_i = e p_i, \dot{p}_0 = 0, \dot{p}_i = 0, p_0^2 - p_i^2 - m^2 = 0$.

At this juncture, it is worthwhile to lay emphasis on the crucial fact that the forms of the non-trivial canonical *commutative* brackets $\{x_0, x_i\}_{(PB)} = 0, \{x_i, x_j\}_{(PB)} = 0$ (also $\{p_0, p_i\}_{(PB)} = 0, \{p_i, p_j\}_{(PB)} = 0$) in the untransformed frames (corresponding to the *commutative* geometry of the spacetime) remain intact in the gauge transformed frames

$$\begin{array}{ll} x_0 \rightarrow X_0 = x_0 + \xi p_0, & \quad p_0 \rightarrow P_0 = p_0, \\ x_i \rightarrow X_i = x_i + \xi p_i, & \quad p_i \rightarrow P_i = p_i, \end{array} \tag{2.6}$$

as can be checked clearly by the explicit computations $\{X_0, X_i\}_{(PB)} = 0, \{X_i, X_j\}_{(PB)} = 0$ (and $\{P_0, P_i\}_{(PB)} = 0, \{P_i, P_j\}_{(PB)} = 0$). This demonstrates that the gauge transformations (2.6) generate *no* NC in the spacetime structure. This fact is reflected in the invariance of the Poincaré algebra constructed by the momenta p_μ and the boost $M_{0i} = x_0 p_i - x_i p_0$ and the rotation $M_{ij} = x_i p_j - x_j p_i$ generators as given below:

$$\begin{array}{l} \Big\{p_\mu, p_\nu\Big\}_{(PB)} = 0, \qquad \Big\{M_{\mu\nu}, p_\lambda\Big\}_{(PB)} = \delta_{\mu\lambda} p_\nu - \delta_{\nu\lambda} p_\mu, \\ \Big\{M_{\mu\nu}, M_{\lambda\zeta}\Big\}_{(PB)} = \delta_{\mu\lambda} M_{\nu\zeta} + \delta_{\nu\zeta} M_{\mu\lambda} - \delta_{\mu\zeta} M_{\nu\lambda} - \delta_{\nu\lambda} M_{\mu\zeta}, \end{array} \tag{2.7}$$

where the antisymmetric generator $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ encompasses the boost (M_{0i}) as well as the rotation (M_{ij}) generators. The above invariance exists because of the gauge-invariance (i.e. $p'_\mu \equiv P_\mu = p_\mu, M'_{\mu\nu} \equiv X_\mu P_\nu - X_\mu P_\nu = M_{\mu\nu}$) of the generators p_μ and $M_{\mu\nu}$ which can be checked explicitly by using (2.6). The above algebra should be contrasted against the Poisson brackets between the spacetime variables x_μ (which are *not* gauge-invariant) with the gauge-invariant Poincaré generators p_μ and $M_{\mu\nu}$. For instance, it can be checked, using (2.6), that $\{x_\mu, p_\nu\}_{(PB)} = \delta_{\mu\nu} \rightarrow \{X_\mu, P_\nu\}_{(PB)} = \{x_\mu, p_\nu\}_{(PB)} = \delta_{\mu\nu}$

[§] The generator G for the gauge transformations (2.5) can be written in terms of the first-class constraints as: $G = \dot{\xi}\Pi_e + (\xi/2)(p_0^2 - p_i^2 - m^2)$. The gauge transformations δ_g for the generic field ϕ can be written as: $\delta_g \phi = \{\phi, G\}_{(PB)}$ where the canonical Poisson brackets $\{x_0, p_0\}_{(PB)} = 1, \{x_i, p_j\}_{(PB)} = \delta_{ij}, \{e, \Pi_e\}_{(PB)} = 1$ (taking into account the rest of the brackets to be zero) have to be exploited for the explicit derivations.

showing that the algebra between gauge non-invariant spacetime variable x_μ and the gauge-invariant momentum generator p_μ remain *invariant*. However, the algebra between the gauge-invariant angular momenta $M_{\mu\nu}$ (i.e. $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \to M'_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu = M_{\mu\nu}$) and the gauge non-invariant spacetime variables x_μ (i.e. $x_\mu \to X_\mu = x_\mu + \xi\, p_\mu$):

$$\big\{M_{\mu\nu}, x_\lambda\big\}_{(PB)} = \delta_{\mu\lambda} x_\nu - \delta_{\nu\lambda} x_\mu \;\to\; \big\{M_{\mu\nu}, X_\lambda\big\}_{(PB)} = \delta_{\mu\lambda} X_\nu - \delta_{\nu\lambda} X_\mu, \tag{2.8}$$

remains *form-invariant* where X_μ is the gauge-transformed variable defined in (2.6).

Now we shall dwell a bit on the explicit derivation of the infinitesimal gauge symmetry transformations (2.5) by requiring the mutual consistency among (i) the equations of motion derived from the set of equivalent Lagrangians (2.1), (ii) the definition of the canonical momenta p_μ from these Lagrangians, and (iii) the basic gauge symmetry transformations (i.e. $\delta_g x_0 = \xi p_0, \delta_g x_i = \xi p_i$) for the space and time variables x_i and x_0. For instance, it is clear from the equation (2.2) that $\delta_g e = (1/m)\delta_g\{[\dot x_0^2 - \dot x_i^2]^{(1/2)}\}$. Exploiting the basic gauge transformations for the spacetime variables ($\delta_g x_\mu = \xi p_\mu$) and using the equations of motion $\dot p_\mu = 0$ as well as the definition of p_μ from (2.2), it can be checked that $\delta_g e = \dot\xi$. In a similar fashion, using the gauge transformations on the equations of motion $\dot x_\mu = e p_\mu$ (i.e. $\delta_g \dot x_\mu = \delta_g\ [e p_\mu]$), it is obvious that $\delta_g p_0 = 0, \delta_g p_i = 0$ if we exploit the equation of motion $\dot p_\mu = 0$ and earlier derived gauge transformation (i.e. $\delta_g e = \dot\xi$) for the einbein field e. Alternatively, the equation of motion $\dot x_\mu = e p_\mu$ implies that $p_\mu = (\dot x_\mu / e)$. Exploiting the earlier derived gauge transformation $\delta_g e = \dot\xi$ and the basic gauge transformations $\delta_g x_\mu = \xi p_\mu$, we obtain

$$\delta_g p_\mu = \frac{\dot\xi}{e}\,\Big(p_\mu - \frac{\dot x_\mu}{e}\Big) + \frac{\xi}{e}\,\dot p_\mu, \tag{2.9}$$

which ultimately implies: $\delta_g p_\mu = 0$ if we use the equations of motion $\dot p_\mu = 0, \dot x_\mu = e p_\mu$. So far, the above gauge transformations have not been applied to the equations of motion $\dot p_\mu = 0, p_0^2 - p_i^2 - m^2 = 0$ derived from (2.3). In fact, the explicit gauge transformations (i.e. $\delta_g x_0 = \xi p_0, \delta_g x_i = \xi p_i, \delta_g p_0 = 0, \delta_g p_i = 0, \delta_g e = \dot\xi$) are consistent with the equations of motions (i.e. $\dot p_0 = 0, \dot p_i = 0$ and $p_0^2 - p_i^2 - m^2 = 0$) because $\delta_g \dot p_\mu \equiv (d/d\tau)[\delta_g p_\mu] = 0$ and $\delta_g[p_0^2 - p_i^2 - m^2 = 0] \Rightarrow p_0 \delta_g p_0 - p_i \delta_g p_i = 0$ are trivially satisfied. This technique of the derivation of the gauge transformations for the rest of the fields, from a *given* symmetry transformation for the basic spacetime variables, will be exploited in the next section.

3 Noncommutativity and Non-Standard Symmetries

Similar to the standard gauge transformations (2.6), let us pay attention to the following non-standard spacetime transformations (with an infinitesimal parameter $\zeta(\tau)$)

$$\begin{array}{l} x_0 \to X_0 = x_0 + \zeta\ \theta_{0i}\ p_i \;\Rightarrow\; \tilde\delta_g x_0 = \zeta\ \theta_{0i}\ p_i, \\ x_i \to X_i = x_i + \zeta\ \theta_{i0}\ p_0 \;\Rightarrow\; \tilde\delta_g x_i = \zeta\ \theta_{i0}\ p_0, \end{array} \tag{3.1}$$

which imply a time-space NC because the nontrivial Poisson bracket for the transformed spacetime variables turns out to be non-zero (i.e. $\{X_0, X_i\}_{(PB)} = -2\zeta\theta_{0i}$). In the above

derivation, we have (i) treated the antisymmetric (i.e. $\theta_{0i} = -\theta_{i0}$) parameter θ_{0i} to be a constant (i.e. independent of the parameter τ as well as the phase space variables), and (ii) exploited the canonical brackets $\{x_\mu, x_\nu\}_{(PB)} = 0, \{p_\mu, p_\nu\}_{(PB)} = 0, \{x_\mu, p_\nu\}_{(PB)} = \delta_{\mu\nu}$. It is elementary to check that $\{X_0, X_0\}_{(PB)} = \{X_i, X_j\}_{(PB)} = 0$. One can treat the above NC to be a special case of the general NC defined through $\{X_\mu(\tau), X_\nu(\tau)\}_{(PB)} = \Theta_{\mu\nu}(\tau)$ on the spacetime target manifold where $\Theta_{0i}(\tau) = -2\zeta(\tau)\theta_{0i}, \Theta_{ij}(\tau) = -2\zeta(\tau)\theta_{ij} = 0$ ¶. In fact, such a kind of NC has been discussed extensively in [15-17]. The special type of transformations (3.1) have been taken into account primarily for a couple of reasons. First, they lead to the time-space NC (i.e. $\theta_{0i} \neq 0, \theta_{ij} = 0$) in the transformed spacetime manifold which has been used, in detail, for the development of a unitary quantum mechanics [15]. Second, they are relevant in the context of BRST symmetry transformations and corresponding cohomology (see, e.g., Section 5 below for a detailed discussion).

Taking into account the basic transformations (3.1) on the spacetime variables and demanding their consistency with some of the equations of motion, derived from the set of Lagrangians (2.1), we obtain (using the trick discussed earlier in connection with the derivation of the standard gauge transformations) the following non-standard transformations for the rest of the dynamical variables of the Lagrangian L_f of (2.3), namely;

$$\begin{aligned} \tilde{\delta}_g e &= \frac{2\,\dot{\zeta}\,\theta_{0i}\,p_0\,p_i}{m^2} \equiv \frac{2\,\dot{\zeta}\,\theta_{0i}\,\dot{x}_0\,\dot{x}_i}{\dot{x}^2}, \\ \tilde{\delta}_g p_0 &= \frac{2\,\dot{\zeta}\,\theta_{0i}\,p_i}{e}\,\Big[\,\frac{1}{2} - \frac{p_0^2}{m^2}\,\Big], \\ \tilde{\delta}_g p_i &= -\frac{2\,\theta_{0j}\,p_0\,\dot{\zeta}}{e}\,\Big[\,\frac{\delta_{ij}}{2} + \frac{p_i p_j}{m^2}\,\Big]. \end{aligned} \tag{3.2}$$

It should be noted that, so far, the above transformations have *not* been applied to (and contrasted against) the sanctity of the equations of motion $\dot{p}_0 = 0, \dot{p}_i = 0, p_0^2 - p_i^2 - m^2 = 0$. It can be checked that the above non-standard transformations are consistent with $p_0^2 - p_i^2 = m^2$ because the relation $\tilde{\delta}_g p_0 = (1/p_0)[p_i \tilde{\delta}_g p_i]$ is satisfied without any restriction on any parameters. However, the story is totally different as far as the consistency with $\tilde{\delta}_g \dot{p}_\mu = 0$ is concerned. It can be seen that, on the on-shell $\dot{p}_0 = 0, \dot{p}_i = 0$, we have

$$\frac{d}{d\tau}(\tilde{\delta}_g p_0) = 0 \Rightarrow \frac{2}{e^2}\,\Big[\ddot{\zeta} e - \dot{\zeta}\dot{e}\Big]\,\Big[\theta_{0i}\,p_i\Big(\frac{1}{2} - \frac{p_0^2}{m^2}\Big)\Big] = 0, \tag{3.3}$$

which can be satisfied (for $e \neq 0$) by any (or all) of the following conditions

$$(i)\;\; \theta_{0i} p_i = 0, \qquad (ii)\;\; \ddot{\zeta} e - \dot{\zeta}\dot{e} = 0, \qquad (iii)\;\; p_0^2 = m^2/2. \tag{3.4}$$

It is evident that $\theta_{0i} p_i = 0$ is *not* an interesting solution because, in this case, there is no transformation (i.e. $\tilde{\delta}_g x_0 = \theta_{0i} p_i = 0$) for the time variable x_0. As a result, there will be no

¶We shall furnish, later in the present section, a proof of $\theta_{ij} = 0 (\Rightarrow \Theta_{ij} = 0)$ by taking into account the consistency between the symmetry properties and the equations motion for the model under consideration.

time-space NC in the theory. Before we shall focus on the solutions (ii) and (iii) of (3.4), let us find out the restrictions from the following (i.e. $\tilde{\delta}_g[\dot{p}_i] = 0$) consistency condition

$$\frac{d}{d\tau}(\tilde{\delta}_g p_i) = 0 \Rightarrow -\frac{2}{e^2}\,\Big[\ddot{\zeta} e - \dot{\zeta}\dot{e}\Big]\,\Big[\theta_{0j}\;p_0\Big(\frac{\delta_{ij}}{2} + \frac{p_i p_j}{m^2}\Big)\Big] = 0. \tag{3.5}$$

It is clear that, for $e \neq 0$, the above restrictions are satisfied by the solution (ii) of (3.4) and, in addition, there is yet another restriction as given below

$$\frac{\theta_{0i} p_0}{2} + p_i\,\frac{\theta_{0j} p_0 p_j}{m^2} = 0. \tag{3.6}$$

It is evident now that the conditions to be satisfied are (3.6) as well as (ii) and (iii) of (3.4) so that the restrictions (3.3) and (3.5) could be satisfied *together*.

Let us now embark on the explicit solutions to (ii) of equation (3.4) which is a common condition for the solution to (3.3) and (3.5). As a side remark, it is worthwhile to mention that such a condition has also been obtained in our earlier work [22] on reparametrization invariant toy model of a non-relativistic particle. We recapitulate here some of the key points of our arguments in [22]. It is evident, from the condition (ii) of (3.4), that $(\ddot{\zeta}/\dot{\zeta}) = (\dot{e}/e) = R$ where the ratio R is some well-defined function of τ (or a constant). Let us choose R to be a constant: $R = \pm K$ where K is independent of τ. The solutions that emerge for e and ζ are: $e(\tau) = e(0)e^{\pm K\tau}, \zeta(\tau) = \zeta(0)e^{\pm K\tau}$. Unfortunately, these values, when substituted in (3.2), lead to no interesting symmetry transformations for the first-order Lagrangian L_f. Further, a simpler non-trivial choice $R = \pm\tau$ leads to the solutions for e and ζ as: $e(\tau) = e(0)e^{\pm(\tau^2/2)}$ and a series solution $\zeta(\tau) = \tau \pm \sum_{n=1}^{\infty}\,[(2n-1)!!]/[(2n+1)!]\;\tau^{2n+1}$. The substitution of these values in (3.2), once again, does not lead to any worthwhile symmetry property of L_f. Thus, we are led to conclude that, even though, the restriction (ii) of (3.4), is a common solution to (3.3) and (3.5), its thorough discussion does not lead to any interesting symmetry property of L_f. To be precise, we do not know, at the moment, any worthwhile solution to the condition (ii) of (3.4) that leads to some interesting symmetry property of L_f. At present, it is an open problem to find out an interesting solution to (ii) of equation (3.4) (which happens to be one of the solutions to (3.5), too.)

We shall now focus on the left over conditions $p_0^2 = (m^2/2)$ and (3.6). The former one trivially implies that $p_0 = \pm(m/\sqrt{2})$. However, to obtain the solution to the latter one, our previous knowledge of our earlier work [22] on the reparametrization invariant free non-relativistic particle turns out to be quite handy. It is clear that if we make the following choice in (3.6)

$$\theta_{0i}\;p_0\;p_i = \frac{m^2}{2}, \tag{3.7}$$

a couple of interesting consequences emerge very naturally. First, we notice that the transformation on the einbein field becomes the gauge transformation (i.e. $\tilde{\delta}_g \to \delta_g$) of (2.5) with the identification $\zeta(\tau) = \xi(\tau)$); namely

$$\tilde{\delta}_g\;e = \frac{2\;\dot{\zeta}\;\theta_{0i}\;p_0\;p_i}{m^2} \quad\rightarrow\quad \delta_g\;e = \dot{\zeta} \equiv \dot{\xi}. \tag{3.8}$$

Second, the relation in (3.6) with the above choice, ultimately, leads to (i) a connection between p_0 and p_i through noncommutative parameter θ_{0i}, and (ii) the noncommutative parameters θ_{0i} are restricted. These observations are captured in the following relations:

$$p_i = \theta_{i0} p_0, \qquad \theta_{0i}\theta_{0j} = \theta_{i0}\theta_{j0} = -\delta_{ij}, \qquad p_0 = \theta_{0i} p_i, \qquad p_i p_j = -\frac{m^2}{2}\delta_{ij}. \tag{3.9}$$

It is now obvious that the non-standard transformations (3.1) and (3.2) reduce *exactly* to a particular case of the standard gauge transformations (2.5). However, there is a catch. The zeroth component of the momentum has now been fixed (i.e. $p_0 = +(m/\sqrt{2})$) ‖. As a consequence, the space component of momentum is also fixed as: $p_i = (m/\sqrt{2})\,\theta_{i0}$. Thus, an interesting limiting case (i.e. $\delta_g \to \delta_g^{(sp)}$) of the gauge transformations (2.5) emerges, from the non-standard transformations (3.1) and (3.2) (i.e. $\tilde{\delta}_g \to \delta_g^{(sp)}$ with the identification ($\xi(\tau) = \zeta(\tau)$), namely;

$$\delta_g^{(sp)} x_0 = \frac{m}{\sqrt{2}}\xi, \quad \delta_g^{(sp)} x_i = \frac{m}{\sqrt{2}}\xi\theta_{i0}, \quad \delta_g^{(sp)} p_0 = 0, \quad \delta_g^{(sp)} p_i = 0, \quad \delta_g^{(sp)} e = \dot{\xi}. \tag{3.10}$$

At this juncture, there are a few noteworthy points that have to be mentioned. First of all, we note that $\delta_g^{(sp)}$ emerges from (2.5) as well as from (3.1) and (3.2). Thus, there are a couple of ways to interpret the symmetry transformations (3.10). Second, the values for $p_0 = (m/\sqrt{2})$ and $p_i = (m/\sqrt{2})\,\theta_{i0}$, obtained from the consistency requirements, satisfy all the equations of motion (i.e. $\dot{p}_0 = 0, \dot{p}_i = 0, p_0^2 - p_i^2 = m^2$) and the choice (3.7) with inputs from (3.9). Third, with the above values of p_0 and p_i, the first-order Lagrangian L_f in (2.3) reduces effectively to the following simpler looking form

$$L_{f1}^{(eff)} = \frac{m}{\sqrt{2}}\,\dot{x}_0 - \frac{m}{\sqrt{2}}\,\dot{x}_i\,\theta_{i0}. \tag{3.11}$$

The above Lagrangian remains quasi-invariant (i.e. $\delta_g^{(sp)} L_{f1}^{(eff)} = (d/d\tau)\,[m^2\xi]$) under the symmetry transformations (3.10). There is yet another way to obtain a simpler form of (2.3) (with the inputs $p_0 = (m/\sqrt{2})$ and $p_i = (m/\sqrt{2})\theta_{i0}$) because now $\dot{x}_0 = e p_0 \equiv (m/\sqrt{2})\,e$ and $\dot{x}_i = (m/\sqrt{2})\,e\,\theta_{i0}$ imply that the above effective Lagrangian (3.11) can be re-expressed as: $L_{f2}^{(eff)} = m^2\,e$. It is evident that, under the transformations (3.10), the effective Lagrangian $L_{f2}^{(eff)}$ also remains quasi-invariant because $\delta_g^{(sp)} L_{f2}^{(eff)} = (d/d\tau)\,[m^2\xi]$. It is very interesting to note that our starting Lagrangian $L_0 = m(\dot{x}^2)^{(1/2)}$ also transforms to $\delta_g L_0 = (d/d\tau)\,[m^2\xi]$ under the usual gauge transformation $\delta_g x_\mu = \xi\,[m\dot{x}_\mu/(\dot{x}^2)^{(1/2)}]$.

Now we shall dwell a bit on the NC and commutativity properties of the present model of the massive relativistic particle. It has been shown in [18,19], through Dirac bracket formalism, that the NC and commutativity for this model owe their origin to the different choices of the gauge-condition. These gauge-conditions, in turn, are connected to each-other by some kind of a gauge transformation. Thus, the NC and commutativity of this model, in

‖The choice $p_0 = -(m/\sqrt{2})$, implying $p_i = -(m/\sqrt{2})\theta_{i0}$, does also lead to a symmetry transformation. However, the negative energy ($p_0 = -(m/\sqrt{2})$ solution is not a physically interesting choice for a "particle".

some sense, are equivalent (because one can gauge away the NC by the redefinition of the variables [18]). This result can be explained in the language of symmetry properties of the Lagrangian L_f. It is clear from the gauge transformations (2.6) that there is no spacetime NC in the (un-)transformed frames. Thus, equation (3.10), which can be derived from the gauge transformation (2.5) with the substitution $p_0 = (m/\sqrt{2})$ and $p_i = (m/\sqrt{2})\,\theta_{i0}$, implies *no* NC at all. However, if we focus on (3.10) from the point of view of (3.1) and (3.2) (with conditions listed in (3.9)), we find that there exists a NC in the transformed frames of spacetime which entails upon the mass parameter m to become *noncommutative* in nature. Let us recall that, the NC present in the transformed frames (i.e. $\{X_0, X_i\}_{(PB)} = -2\zeta\theta_{0i}$) corresponding to the transformations (3.1), basically owes its origin to the canonical brackets $\{x_0, p_0\}_{(PB)} = 1, \{x_i, p_j\}_{(PB)} = \delta_{ij}$. Thus, it is clear that, with the solutions $p_0 = (m/\sqrt{2}), p_i = (m/\sqrt{2})\theta_{i0}, \theta_{0i}\theta_{0j} = -\delta_{ij}$, these brackets reduce to

$$\Big\{x_0, m\Big\}_{(PB)} = \sqrt{2}, \qquad \Big\{x_i, m\Big\}_{(PB)} = -\sqrt{2}\,\theta_{i0}, \tag{3.12}$$

demonstrating that the original NC associated with the transformations (3.1) enforces the mass parameter to become noncommutative. It is interesting to emphasize that the NC of the mass parameter m with the time (i.e. x_0) and the space (i.e. x_i) variables is primarily due to the connections (i.e. $p_i = \theta_{i0}p_0, p_0 = \theta_{0i}\;p_i$) between the momentum components p_0 and p_i through the antisymmetric parameter θ_{0i}. Thus, the above arguments establish that the NC and commutativity of the model under consideration are equivalent and are different facets of a specific gauge symmetry transformation (cf. (3.10)). The NC of mass parameter is *not* a new observation. In the realm of quantum groups, the NC nature of mass parameter has been shown in the context of free motion of the (non-)relativistic particles on a quantum-line, embedded in a cotangent manifold [25-28].

Before we wrap up this section, we would like to shed some light on the condition $\theta_{ij} = 0$ from the requirement of the consistency between the symmetry properties and the equations of motion for the model under consideration. Towards this goal in mind, let us begin with a more general non-standard spacetime transformation $\tilde{\delta}_g^{(1)}$

$$\tilde{\delta}_g^{(1)} x_0 = \zeta_1\;\theta_{0i}\;p_i, \qquad \tilde{\delta}_g^{(1)} x_i = \zeta_1\;(\theta_{i0}\;p_0 + \theta_{ij}\;p_j), \tag{3.13}$$

where $\zeta_1(\tau)$ is an infinitesimal transformation parameter and $\theta_{ij} \neq 0$ so that $\{X_i, X_j\}_{(PB)} = -2\zeta_1\theta_{ij}$. It is elementary to check that, in the limit $\theta_{ij} = 0$, we get back the non-standard transformations (3.1). Given the above spacetime transformations, we have to derive the transformations for the rest of the variables of the Lagrangian L_f (cf. (2.3)). We adopt here the same technique as we have exploited for the derivation of (2.5) and (3.2). These explicit transformations, which are the analogue of (3.2), are as follows:

$$\begin{aligned}
\tilde{\delta}_g^{(1)} e &= \frac{2\;\dot{\zeta}_1\;\theta_{0i}\;p_0\;p_i}{m^2} \equiv \frac{2\;\dot{\zeta}_1\;\theta_{0i}\;\dot{x}_0\;\dot{x}_i}{\dot{x}^2}, \\
\tilde{\delta}_g^{(1)} p_0 &= \frac{2\;\dot{\zeta}_1\;\theta_{0i}\;p_i}{e}\;\Big[\;\frac{1}{2} - \frac{p_0^2}{m^2}\;\Big], \\
\tilde{\delta}_g^{(1)} p_i &= -\frac{2\;\theta_{0j}\;p_0\;\dot{\zeta}_1}{e}\;\Big[\;\frac{\delta_{ij}}{2} + \frac{p_i p_j}{m^2}\;\Big] + \frac{\dot{\zeta}_1\;\theta_{ij}\;p_j}{e}.
\end{aligned} \tag{3.14}$$

The above transformations are, once again, consistent with the equation of motion $p_0^2 - p_i^2 - m^2 = 0$ because it can be checked that $\tilde{\delta}_g^{(1)} p_0 = (1/p_0)\ [p_i \tilde{\delta}_g^{(1)} p_i]$ is satisfied without any restriction on any of the parameters or fields of the theory. However, the consistency with $\tilde{\delta}_g^{(1)} \dot{p}_0 = (d/d\tau)[\tilde{\delta}_g^{(1)} p_0] = 0$ and $\tilde{\delta}_g^{(1)} \dot{p}_i = (d/d\tau)[\tilde{\delta}_g^{(1)} p_i] = 0$ leads to the following additional restriction besides the restrictions given in (3.3) and (3.4), namely;

$$\frac{d}{d\tau}(\tilde{\delta}_g^{(1)} p_i) = 0 \Rightarrow -\frac{2}{e^2}\left[\ddot{\zeta}_1 e - \dot{\zeta}_1 \dot{e}\right]\left[\theta_{0j}\, p_0\Big(\frac{\delta_{ij}}{2} + \frac{p_i p_j}{m^2}\Big) - \frac{\theta_{ij} p_j}{2}\right] = 0, \tag{3.15}$$

which is nothing but the generalization of (3.5). Analogous discussions, as given earlier, demonstrate that the generalized version of (3.6) is now

$$\frac{\theta_{0i} p_0}{2} + p_i\ \frac{\theta_{0j} p_0 p_j}{m^2} = \frac{\theta_{ij} p_j}{2}. \tag{3.16}$$

A couple of points are clear from the validity of (3.4), even in the present case of our discussions, where $\theta_{ij} \neq 0$. First, the solution (i) of (3.4) is not allowed as argued earlier and the condition (ii) of (3.4) does not lead to any interesting symmetry properties of L_f. Second, the choice $p_0 = (m/\sqrt{2})$ is still an interesting solution emerging from (3.4). Our aim is to obtain a gauge-type transformation from (3.14), too. For this purpose, we summon our knowledge of the previous work [22] and make the choice as given in (3.7) so that we can obtain the usual gauge transformation ($\delta_g e = \dot{\xi}$) for the einbein field $e(\tau)$ with the identification $\zeta_1(\tau) = \xi(\tau)$. The substitution of the values from (3.4) and (3.7), into the condition (3.16), leads to $\theta_{ij} p_j = 0$. For the generality of the transformations, it is proper to assume that $p_i \neq 0$. Thus, a logical conclusion is $\theta_{ij} = 0$ which has been taken into account for the discussion of the unitary quantum mechanics [15]. This establishes the fact that the consistency requirement between the equations of motion and the exact symmetry properties enforces the NC parameter in $\{X_\mu(\tau), X_\nu(\tau)\}_{(PB)} = \Theta_{\mu\nu}(\tau)$ (with $\Theta_{\mu\nu}(\tau) = -2\zeta(\tau)\theta_{\mu\nu}$) to possess only the component $\Theta_{0i}(\tau) = -2\zeta(\tau)\theta_{0i}$ and the component $\Theta_{ij} = -2\zeta(\tau)\theta_{ij}$ is zero due to $\theta_{ij} = 0$. It is now obvious that the transformations (3.1) and (3.2), with certain restrictions emerging from the consistency conditions between the equations of motion and symmetry properties, are the only allowed transformations for our present model.

4 Deformations of the Algebras

Let us now concentrate on the usual Poincaré algebra (2.7) and an additional algebra (2.8) between the angular momentum generator ($M_{\mu\nu}$) and the spacetime variable (x_μ). As mentioned earlier, the angular momentum ($M_{\mu\nu}$) and momentum (p_μ) generators remain invariant under the usual gauge transformations (2.6) (or (2.5)). In other words, these generators and their algebra remain *invariant* in the (un-)transformed frames. This statement is true even with the transformations given in (3.10) which have been obtained from the non-standard gauge-type symmetry transformations in (3.1) and (3.2). However, the story is quite different with the additional algebra (2.8). To observe the impact of NC on this

algebra, let us express the algebra (2.8) in the component form with the boost generator M_{0i} and the rotation generator M_{ij}. The set of these Poisson brackets, in explicit form, are

$$\begin{aligned} \{M_{0i}, x_j\}_{(PB)} &= -\delta_{ij} x_0, & \{M_{0i}, x_0\}_{(PB)} &= x_i, \\ \{M_{ij}, x_0\}_{(PB)} &= 0, & \{M_{ij}, x_k\}_{(PB)} &= \delta_{ik} x_j - \delta_{jk} x_i. \end{aligned} \tag{4.1}$$

It is evident, from (2.8) itself, that the algebra (4.1) remains *form-invariant* in the gauge-transformed frame where $x_0 \to X_0, x_i \to X_i$ (cf. (2.6)) and $M_{0i} \to M_{0i}, M_{ij} \to M_{ij}$. Thus, in the transformed frame, the above algebra will be replaced by an algebra where $x_0 \to X_0$ and $x_i \to X_i$ (cf. (2.6)). In the noncommutative case, corresponding to the transformations (3.10), the transformed spacetime variables $(X_i^{(sp)}, X_0^{(sp)})$ and the gauge invariant boost (M_{0i}) and rotation (M_{ij}) generators are

$$\begin{aligned} x_0 \to X_0^{(sp)} &= x_0 + \xi \frac{m}{\sqrt{2}}, & x_i \to X_i^{(sp)} &= x_i + \xi \theta_{i0} \frac{m}{\sqrt{2}}, \\ M_{0i} &= x_0 \theta_{i0} \frac{m}{\sqrt{2}} - x_i \frac{m}{\sqrt{2}}, & M_{ij} &= x_i \theta_{j0} \frac{m}{\sqrt{2}} - x_j \theta_{i0} \frac{m}{\sqrt{2}}, \end{aligned} \tag{4.2}$$

where we have used $p_0 = (m/\sqrt{2})$ and $p_i = \theta_{i0}(m/\sqrt{2})$. With the inputs from (4.2), it can be checked that the algebra (4.1) (in the component form) changes to the following:

$$\begin{aligned} \{M_{0i}, x_j\}_{(PB)} &= -\delta_{ij} x_0 - x_i \theta_{j0}, & \{M_{0i}, x_0\}_{(PB)} &= x_i - \theta_{i0} x_0, \\ \{M_{ij}, x_0\}_{(PB)} &= x_j \theta_{i0} - x_i \theta_{j0}, & \{M_{ij}, x_k\}_{(PB)} &= \delta_{ik} x_j - \delta_{jk} x_i. \end{aligned} \tag{4.3}$$

There are a few comments in order now. First, it can be seen that, in the limit $\theta_{i0} \to 0$, we do get back the algebra (4.1) for the usual commutative spacetime. Second, we have exploited the NC of the mass parameter through the Poisson brackets (3.12) (and their off-shoots $\{m, x_0\}_{(PB)} = -\sqrt{2}, \{m, x_i\}_{(PB)} = +\sqrt{2}\theta_{i0}$). Third, it is the NC of the mass parameter m with *both* the space x_i and time x_0 variables (cf. (3.12)) which is the root cause of the deformation of the algebra (4.1) to (4.3). Fourth, we have exploited appropriately (3.9) in the above computation of the Poisson brackets (4.3) so that its *form* could be compared with (and contrasted against) (4.1). Fifth, it can be checked that, in the transformed frames (with spacetime variables $X_i^{(sp)}$ and $X_0^{(sp)}$), the algebra (4.3) remains *form-invariant* in the sense that one has to merely replace: $x_0 \to X_0^{(sp)}, x_i \to X_i^{(sp)}$.

Now let us pay attention to the impact of the NC on the modification of Poincaré algebra (2.7). It is straightforward to note that the algebra $\{p_\mu, p_\nu\}_{(PB)} = 0$ remains intact with the choice $p_0 = (m/\sqrt{2}), p_i = (m/\sqrt{2})\theta_{i0}$. However, there is some modification (due to the NC in spacetime) in the algebra between the momentum generator p_μ and the angular momentum $M_{\mu\nu}$ when they are written in the component form. For instance, using the expression for the boost and rotation generators from (4.2), it can be checked that

$$\begin{aligned} \{M_{0i}, p_j\}_{(PB)} &= -2\delta_{ij} p_0, & \{M_{0i}, p_0\}_{(PB)} &= 2p_i, \\ \{M_{ij}, p_0\}_{(PB)} &= \theta_{i0} p_j - \theta_{j0} p_i \equiv 0, & \{M_{ij}, p_k\}_{(PB)} &= \delta_{ik} p_j - \delta_{jk} p_i. \end{aligned} \tag{4.4}$$

It should be noted that the r.h.s. (i.e. $\theta_{i0} p_j - \theta_{j0} p_i$) of the bracket $\{M_{ij}, p_0\}_{(PB)}$ can be proved to be zero if we take into account the inputs (i.e. $p_i = \theta_{i0}(m/\sqrt{2}), \theta_{i0}\theta_{j0} = -\delta_{ij}$) from (3.9). In the usual case of the commutative spacetime, the algebra corresponding to (4.4), that emerges from (2.7), is as follows:

$$\begin{array}{ll} \big\{M_{0i}, p_j\big\}_{(PB)} = -\delta_{ij} p_0, & \big\{M_{0i}, p_0\big\}_{(PB)} = p_i, \\ \big\{M_{ij}, p_0\big\}_{(PB)} = 0, & \big\{M_{ij}, p_k\big\}_{(PB)} = \delta_{ik} p_j - \delta_{jk} p_i. \end{array} \tag{4.5}$$

At this juncture, a couple of remarks are in order. First, the root cause of the difference of factor two, in the above equations (4.4) and (4.5), is the NC of the mass parameter m with both the space x_i and time x_0 variables. This is why, a factor of two appears in the Poisson brackets wherever the boost (M_{0i}) generators are present. This factor does not turn up in the Poisson brackets with the rotation generator M_{ij}. Second, the Poisson brackets (3.12) (and their antisymmetric versions), together with the conditions (3.9), have been appropriately exploited to express the structure of (4.4) so that it could be compared with the usual form of algebra in (4.5). The triplet of the Poisson brackets among the boost M_{0i} and rotation M_{ij} generators, that emerges from the usual Poincaré algebra (2.7), are

$$\begin{array}{l} \Big\{M_{ij}, M_{kl}\Big\}_{(PB)} = \delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} - \delta_{jk} M_{il}, \\ \Big\{M_{ij}, M_{0k}\Big\}_{(PB)} = \delta_{ik} M_{0j} - \delta_{jk} M_{0i}, \qquad \Big\{M_{0i}, M_{0j}\Big\}_{(PB)} = M_{ij}. \end{array} \tag{4.6}$$

Let us now compute the *extra* contribution to the above algebra when the NC of spacetime is taken into account. For this purpose, we have to plug in the exact expressions for the boost and rotation generators from (4.2). The explicit computation of the Poisson brackets, using (3.12) and its antisymmetric versions, leads to the following:

$$\begin{array}{l} \Big\{M_{ij}, M_{kl}\Big\}_{(PB)} = \delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} - \delta_{jk} M_{il}, \\ \Big\{M_{ij}, M_{0k}\Big\}_{(PB)} = 2\,\Big[\,\delta_{ik} M_{0j} - \delta_{jk} M_{0i}\,\Big], \qquad \Big\{M_{0i}, M_{0j}\Big\}_{(PB)} = 2 M_{ij}. \end{array} \tag{4.7}$$

It is clear, once again, that the extra factor of 2 appears in the case of the Poisson brackets where the boost generators M_{0i} are present (and the NC of the mass parameter m with both the space and time variables is taken into account). It is obvious that all the algebras, from (4.4) to (4.7), remain *invariant* in the (un-)transformed frames.

5 Noncommutativity and (Anti-)BRST Symmetries

As discussed earlier, the first-class constraints for the Lagrangian L_f in (2.3), generate a set of "classical" local gauge symmetry transformations (2.5) for all the fields. These "classical" gauge symmetries can be traded with the "quantum" gauge symmetries which are popularly known as the BRST symmetries. These are the generalized "quantum" versions of the local gauge symmetries and are found to be nilpotent of order two. In the BRST formalism, the unitarity and the "quantum" gauge (i.e. BRST) invariance are

respected together at any arbitrary order of perturbative computations for a given physical process in a BRST invariant gauge theory [29]. The (anti-)BRST invariant version of the local gauge-invariant Lagrangian (2.3) is (see, e.g., [24])

$$L_b = p_0 \dot{x}_0 - p_i \dot{x}_i - \tfrac{1}{2}\, e\, (p_0^2 - p_i^2 - m^2) + b\, \dot{e} + \tfrac{1}{2}\, b^2 - i\, \dot{\bar{c}}\, \dot{c}, \tag{5.1}$$

where b is the Nakanishi-Lautrup auxiliary field and $(\bar{c})c$ are the anticommuting (i.e. $c^2 = \bar{c}^2 = 0, c\bar{c} + \bar{c}c = 0$) (anti-)ghost fields which are required in the theory to maintain the unitarity (see, e.g., [29] for details). The above Lagrangian L_b remains quasi-invariant under the following off-shell nilpotent ($s_{(a)b}^2 = 0$) and anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-)BRST transformations $s_{(a)b}$ **

$$\begin{array}{llll} s_b x_0 = c p_0, & s_b x_i = c p_i, & s_b p_0 = 0, & s_b p_i = 0, \\ s_b c = 0, & s_b e = \dot{c}, & s_b \bar{c} = i b, & s_b b = 0, \end{array} \tag{5.2}$$

$$\begin{array}{llll} s_{ab} x_0 = \bar{c} p_0, & s_{ab} x_i = \bar{c} p_i, & s_{ab} p_0 = 0, & s_{ab} p_i = 0, \\ s_{ab} \bar{c} = 0, & s_{ab} e = \dot{\bar{c}}, & s_{ab} c = -i b, & s_{ab} b = 0, \end{array} \tag{5.3}$$

which are the "quantum" generalization of the "classical" local gauge transformations (2.5). The above off-shell nilpotent transformations †† are generated by the conserved and off-shell nilpotent ($Q_{(a)b}^2 = 0$) (anti-)BRST charges $Q_{(a)b}$ as given below:

$$Q_b = b\dot{c} + \frac{c}{2}(p_0^2 - p_i^2 - m^2) \equiv b\dot{c} - \dot{b}c, \quad Q_{ab} = b\dot{\bar{c}} + \frac{\bar{c}}{2}(p_0^2 - p_i^2 - m^2) \equiv b\dot{\bar{c}} - \dot{b}\bar{c}, \tag{5.4}$$

because $s_{(a)b}\phi = -i[\phi, Q_{(a)b}]_\pm$ is true for the generic field $\phi = x_0, x_i, p_0, p_i, e$ of the theory. The subscripts $(+)-$ on the square bracket correspond to the (anti-)commutators for the generic field ϕ being (fermionic)bosonic in nature. The physicality criteria $Q_{(a)b}|phys >= 0$, on the physical states of the total Hilbert space, imply that the real physical states $|phys>$ are annihilated by the operator form of the first-class constraints $\Pi_e = b$ and $\dot{b} = -(1/2)(p_0^2 - p_i^2 - m^2)$. In other words, the conditions $\Pi_e|phys >= 0$ and $(p_0^2 - p_i^2 - m^2)|phys >= 0$ are consistent with the Dirac's prescription for the quantization of theories, endowed with the first-class constraints. In physical terms, the primary constraint condition $\Pi_e|phys >= 0 (\Rightarrow b|phys >= 0)$ remains intact with respect to the time evolution of the system because its time derivative (i.e. $\dot{b}|phys >= 0 \Rightarrow (p^2 - m^2)|phys >= 0$) corresponds to the annihilation of the physical states by the secondary constraint ($p^2 - m^2 = 0$).

**We adopt here the notations and conventions used by Weinberg [30]. In fact, in its totality, the nilpotent ($\delta_{(a)b}^2 = 0$) (anti-)BRST transformations $\delta_{(a)b}$ are product of an anticommuting ($\eta c + c\eta = 0$, etc.) spacetime independent parameter η and $s_{(a)b}$ with $s_{(a)b}^2 = 0$. The (anti-)BRST prescription is to replace the local gauge parameter ξ of the gauge transformation (2.5) by η and the (anti-)ghost fields $(\bar{c})c$.

††The on-shell ($\ddot{c} = \ddot{\bar{c}} = 0$) nilpotent ($\tilde{s}_{(a)b}^2 = 0$) version of the (anti-)BRST transformations $\tilde{s}_{(a)b}$: $\tilde{s}_b x_0 = c p_0, \tilde{s}_b x_i = c p_i, \tilde{s}_b p_0 = 0, \tilde{s}_b p_i = 0, \tilde{s}_b c = 0, \tilde{s}_b e = \dot{c}, \tilde{s}_b \bar{c} = -i\dot{e}$ and $\tilde{s}_{ab} x_0 = \bar{c} p_0, \tilde{s}_{ab} x_i = \bar{c} p_i, \tilde{s}_{ab} p_0 = 0, \tilde{s}_{ab} p_i = 0, \tilde{s}_{ab} \bar{c} = 0, \tilde{s}_{ab} e = \dot{\bar{c}}, \tilde{s}_{ab} c = i\dot{e}$ do exist for the Lagrangian $\tilde{L}_b = p_0 \dot{x}_0 - p_i \dot{x}_i - \frac{1}{2} e (p_0^2 - p_i^2 - m^2) - \frac{1}{2}(\dot{e})^2 - i\dot{\bar{c}}\dot{c}$. These can be derived from their off-shell versions (5.2), (5.3) and (5.1), respectively, by the substitution of the equation of motion $b = -\dot{e}$ emerging from the Lagrangian (5.1).

The nilpotency ($Q_b^2 = 0$) property of the conserved BRST charge Q_b and the physicality criteria ($Q_b|phys >= 0$) are the two key ingredients for the definition of the BRST cohomology. In fact, two physical states $|phys >'$ ($= |phys > +Q_b|\chi >$) and $|phys >$ are said to belong to the same cohomology class (i.e. $Q_b|phys >'= 0 \Leftrightarrow Q_b|phys >= 0$) w.r.t. the conserved and nilpotent BRST charge Q_b if they differ by a BRST exact (i.e. $Q_b|\chi >$) state where $|\chi >$ is any arbitrary non-null state of the quantum Hilbert space. Since the BRST transformations s_b (i.e. $s_b\phi = -i[\phi, Q_b]_\pm$ for the generic field ϕ) imbibe the nilpotency property of Q_b, the cohomologically equivalent transformations can be defined in terms of the nilpotent $s_b^2 = 0$ BRST transformations. For instance, the BRST transformed spacetime variables in (5.2) can be re-expressed as

$$\begin{aligned} x_0 \rightarrow X_0 &= x_0 + c\, p_0 \Rightarrow x_0 \rightarrow X_0 = x_0 + s_b\, [x_0], \\ x_i \rightarrow X_i &= x_i + c\, p_i \Rightarrow x_i \rightarrow X_i = x_i + s_b\, [x_i], \end{aligned} \tag{5.5}$$

which show that the untransformed spacetime physical variables (x_i, x_0) and the transformed spacetime variables (X_i, X_0) belong to the same cohomology class w.r.t. the nilpotent transformations s_b as they differ (with each-other) by a BRST exact transformation. It should be noted that the above transformations do *not* lead to any NC in the spacetime structure because the non-trivial brackets (i.e. $\{X_0, X_i\}_{(PB)} = 0, \{X_i, X_j\}_{(PB)} = 0$), in the transformed frames *and* the corresponding brackets (i.e. $\{x_\mu, x_\nu\}_{(PB)} = 0$) in the untransformed frames, are found to be zero. Let us focus on the specific gauge symmetry transformation (3.10) that has been obtained from the non-standard gauge-type transformations (3.1) and (3.2). It is straightforward to obtain the off-shell as well as the on-shell BRST symmetry transformations corresponding to this specific gauge transformation. All one has to do is to replace the local gauge parameter $\xi(\tau)$ by an anticommuting number and the (anti-)ghost fields $(\bar{c})c$ (dictated by the (anti-)BRST prescription). The nilpotent (anti-)BRST transformations can be written in a similar fashion as given in (5.2) and (5.3). In fact, in the language of the BRST transformations corresponding to (3.10), the transformations on the spacetime variables can explicitly be written as

$$x_0 \rightarrow X_0^{(sp)} = x_0 + \frac{m}{\sqrt{2}}\, c, \qquad x_i \rightarrow X_i^{(sp)} = x_i + \frac{m}{\sqrt{2}}\, \theta_{i0}\, c. \tag{5.6}$$

It is straightforward to check that the non-trivial Poisson-bracket $\{X_0^{(sp)}(\tau), X_i^{(sp)}(\tau)\}_{(PB)} = -2c(\tau)\theta_{i0} \equiv \Theta_{0i}(\tau)$ is non-zero leading to the NC in the spacetime structure. In the above computation, the NC of the mass parameter m has been taken into account and the brackets: $\{x_0, m\}_{(PB)} = \sqrt{2}, \{m, x_i\}_{(PB)} = +\sqrt{2}\ \theta_{i0}$, emerging from equation (3.12), have been exploited. This demonstrates that the NC of the transformations (5.6) and the commutativity of the transformations (5.5) are different aspects of the gauge symmetry transformations. This agrees with the discussions about such an equivalence provided in [19] where the language of the Dirac bracket formalism, for different choices of the gauge-conditions, has been exploited for the discussion of reparametrization invariant theories.

It is obvious that the transformations (3.10) have been obtained primarily from the basic non-standard gauge type of transformations (3.1). It would have been pretty difficult to guess these transformations (i.e. (3.10)) from the usual gauge transformations (2.5). To have a closer look at the NC and commutativity discussed above, let us concentrate on the basic transformations (3.1) and argue in the language of the BRST cohomology. With the identification $\zeta(\tau) = \xi(\tau)$ and the application of the BRST prescription, the transformations (3.1) can be written in the language of the BRST transformations in (5.2), as

$$\begin{array}{l} x_0 \to X_0 = x_0 + \theta_{0i}\, c\, p_i \equiv x_0 + s_b\, [\theta_{0i} x_i], \\ x_i \to X_i = x_i + \theta_{i0}\, c\, p_0 \equiv x_i + s_b\, [\theta_{i0} x_0]. \end{array} \tag{5.7}$$

The above transformations lead to the NC in the spacetime structure because the non-trivial bracket (i.e. $\{X_0, X_i\}_{(PB)} = -2c\theta_{0i}$) is non-zero. Here we have used the basic canonical brackets $\{x_0, p_0\}_{(PB)} = 1, \{x_i, p_j\}_{(PB)} = \delta_{ij}$, etc., and as before, the antisymmetric (i.e. $\theta_{0i} = -\theta_{i0}$) NC parameter is treated as a constant tensor. It is elementary to note that, once again, the spacetime untransformed variables (x_i, x_0) and the transformed variables (X_0, X_i) belong to the same cohomology class w.r.t. the BRST transformations s_b. Thus, it is clear that the NC and commutativity for the reparametrization invariant model for the free massive relativistic particle belong to the same cohomology class w.r.t. the nilpotent BRST transformation s_b. All the above arguments could be repeated with the nilpotent anti-BRST transformations s_{ab} (and corresponding conserved and nilpotent charge Q_{ab}), too, to demonstrate the above equivalence.

6 Conclusions

The emphasis, in our present endeavour, is laid on the continuous symmetry properties of the Lagrangian(s) for the physical system of a free massive relativistic particle and their role in the description of the commutativity and NC of the spacetime structure. In particular, the reparametrization, standard gauge- and non-standard gauge-type symmetries play very important roles in our whole discussion. To be specific, a set of non-standard gauge-type transformations (cf. (3.1)) for the space and time variables has been taken into account to demonstrate the existence of a NC in the spacetime structure. This NC primarily exists in the transformed frames. One of the interesting points in our discussion is the fact that these non-standard transformations, leading to a NC, can be guessed from the corresponding standard gauge transformations which lead to the *commutative* transformed frames. In particular, to obtain the non-standard symmetry transformations (e.g. (3.1)) from the standard gauge symmetry transformations (2.6), the infinitesimal gauge increments in the time and space variables are to be exchanged with each-other. For the present model under consideration, this trick has been performed through the introduction of the noncommutative parameter θ_{0i} (cf. (3.1)) which appears in the context of the time-space NC (i.e. $\{X_0, X_i\}_{(PB)} = -2\zeta\theta_{0i}$). This technique works here in our present endeavour because the model under consideration is nothing but the generalization of our earlier work on the

reparametrization invariant toy model of a non-relativistic particle [22]. The basis for such a trick comes from the fundamentals of the BRST cohomology which is discussed, in detail, in Section 5. At the moment, it appears to us that the above procedure will work for all the reparametrization invariant theories where the equivalence between the commutativity and NC has been established through Dirac bracket procedure for the specific choices of the gauge conditions that lead to the existence of NC and commutativity (see, e.g., [19]).

One of the most interesting features of our present investigation on the reparametrization invariant model of a free massive relativistic particle is the existence of a very specific symmetry transformation (3.10) which can be looked upon in *two* distinctly different ways. First, it is a particular case of the usual gauge transformations (2.5) (corresponding to the *commutative* spacetime) when we put $p_0 = (m/\sqrt{2})$ and $p_i = \theta_{i0}(m/\sqrt{2})$ in the expression for the infinitesimal increments in the time and space variables. Second, as discussed in detail in Section 3, this transformation can be obtained from the non-standard gauge-type transformations (3.1) and (3.2) (corresponding to the *noncommutative* spacetime) if we demand the consistency among (i) the basic non-standard transformations for the time and space variables, (ii) the expressions for the canonical momenta derived from various equivalent Lagrangians for the system, and (iii) the equations of motion derived from the above Lagrangians. The outcome of these requirements leads (i) to enforce the mass parameter of the model to be noncommutative (cf. (3.12)) with both the time and space variables, and (ii) to restrict the antisymmetric (i.e. $\theta_{0i} = -\theta_{i0}$) noncommutative parameter θ_{0i} to obey $\theta_{0i}\theta_{0j} = -\delta_{ij}$. The NC of the mass parameter m, in the context of noncommutative geometry, is *not* a speculative and strange idea. For the free motion of the (non-)relativistic (super-)particles on a "quantum" (super) world-line, it has been shown that the mass parameter could become noncommutative [31, 25-28] in the framework of quantum groups where the NC of the mass parameter is altogether of a different variety than our present discussion (in which the NC is of a kind given by (3.12)).

A noteworthy point in our discussion is a minor deformation of the Poincaré algebra by a constant factor and a major deformation in the algebra between the angular momentum generator $M_{\mu\nu}$ and the spacetime variable x_μ due to the presence of a NC in the spacetime structure (see, e.g., Section 4). In particular, the Poisson brackets that include the boost generator M_{0i} pick up an additional factor of two (cf. (4.4),(4.7)) because of the NC of the mass parameter with both the space and time variables (cf. (3.12)). This observation should be contrasted with (i) the results of [19] where the Dirac brackets, corresponding to the Poincaré algebra, do *not* get any contribution from the NC of spacetime because the noncommutative parameter θ_{0i} does not appear in the algebra, and (ii) the results of [18] where the angular momentum generator itself is modified by an extra term for the closure of the algebra. Furthermore, in [19], the NC parameter θ_{0i} appears only in the transformed frames for the Dirac algebra between the angular momentum generator $M_{\mu\nu}$ and the spacetime variable x_μ but it (i.e. θ_{0i}) does not appear in the untransformed frames exactly for the same Dirac algebra. The above observations should be contrasted with

our discussion where (i) the Poisson bracket algebra remains *form-invariant* in both the (un-)transformed frames, and (ii) the noncommutative parameter θ_{0i} appears in both the transformed as well as untransformed frames (cf. (4.3)).

To obtain a time-space NC in the spacetime structure, our approach is a general one. This claim is valid in the sense that it can be applied to any arbitrary reparametrization invariant theories (because, as stressed earlier, the basis for our trick comes from the basics of the BRST cohomology). Our method can be applied now to the reparametrization invariant model of a free relativistic superparticle that has already been discussed in the framework of the (super) quantum groups $GL_{\sqrt{q}}(1|1)$ and $GL_q(2)$ [26]. It would also be a nice endeavour to find a possible connection between these two different approaches. The generalization of our present work to the study of a reparametrization invariant model of an interacting relativistic particle, with the electromagnetic field in the background, is yet another direction that could be pursued in the future. These are some of the issues that are under investigation and our results would be reported elsewhere [32].

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